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AN ASYMPTOTIC MODEL OF A MEMBRANE AND A STRING†

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A three-dimensional problem in the theory of elasticity of a stressed solid in a domain of small thickness (diameter) is converted into a problem in the theory of membranes (strings). The averaged problems obtained enable one to describe membranes (strings) with inhomogeneities of a size comparable with their thickness. The problems are analysed using the asymptotic method of averaging. © 1998 Elsevier Science Ltd. All rights reserved.

A method of converting a three-dimensional problem in the theory of elasticity for a thin domain into a problem in the theory of plates was described in [1, 2]. A transition from a three-dimensional problem in the theory of elasticity in a domain of small diameter to a problem in the theory of beams was made in [3, 4]. An asymptotic method [5, 6] was used in the investigation which has the advantage that it is applicable both to classical homogeneous plates and beams as well as to inhomogeneous periodic structures with inhomogeneities with dimensions comparable with the thickness (homogeneous plates and beams having inhomogeneities with characteristic dimensions which are significantly greater than the thickness have been considered previously [7, 8]). The application of an asymptotic method to stressed structures was begun in [9–14]: it has been applied to three-dimensional structures in [9, 12], to thin plates in [13] and to beams of small diameter in [14, 15]. In particular, averaged equations for the instability of inhomogeneous plates and beams have been obtained. Problems involving inhomogeneous membranes and strings, in which both the small thickness and the initial stresses play a decisive role, touch directly on problems of the above type. As far as we are aware, the derivation of the equations of inhomogeneous membranes and strings from three-dimensional equations in the theory of elasticity has not been considered previously (membranes with inhomogeneities of a characteristic size which is significantly greater than the thickness have been studied). In this paper, the problem is considered at the level of constructing a formal asymptotic expansion [5]. Limiting problems are obtained which yield the equations of inhomogeneous membranes and strings.

While there is a certain resemblance between the methods which have been used previously [2, 9–16] and those used in this paper, the results obtained do not follow from the results of the above-mentioned papers. This is due to the fact that the "mechanics" of a problem is mainly governed by the actual form of the first few terms of the asymptotic expansion [6] and, to a lesser degree, by the general form of the series. The treatment presented also explains the reason for the "inoperability" of a number of the equilibrium equations from [2, 9–16]; they do not correspond to the theory of plates and beams (these equations are "overlooked" within their framework) but to the theory of membranes and strings.

The issues related to the practical application of inhomogeneous and, in particular, reinforced membranes, are described in [17].

1. A MEMBRANE

We consider a domain Q_ε which is obtained by the periodic repetition of a certain periodic cell (PC) with a periodicity P_ε in the x_1x_2 plane (Fig. 1). The characteristic size of this PC (which is identical to the characteristic thickness of the membrane) is a small quantity ε which is formalized in the form $\varepsilon \to 0$. When $\varepsilon \to 0$, the domain Q_ε contracts to a two-dimensional domain S in the x_1x_2 plane. The stress in the material occupying the domain Q_ε, denoted by σ_{ij}^*, are determined from the solution of a problem in the theory of elasticity. We shall assume that the elasticity constants of the membrane material a_{ijkl} and the initial stresses σ_{ij}^* are of the same order ε^{-1}. It is also assumed that the orders of magnitude are identical (in classical theory, the stiffnesses are neglected compared with the stresses) due to the fact that it is impossible to create initial stresses σ_{ij}^* of a greater order of magnitude when a_{ijkl} in the

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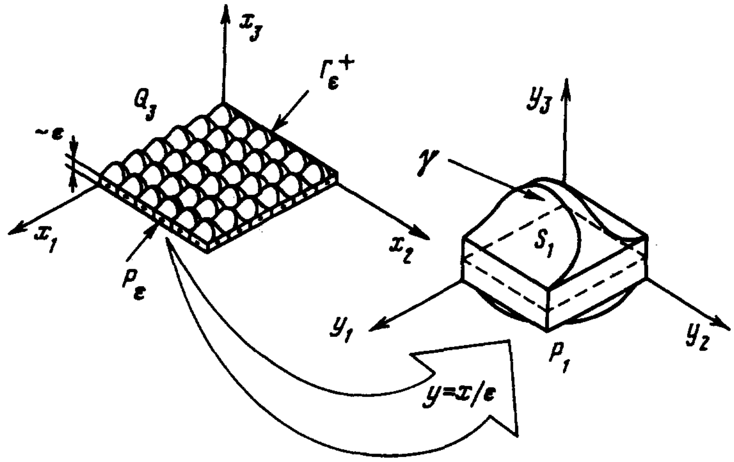

Fig. 1.

case of bounded deformations in the plane of a plate. Then [18], the equilibrium equations for a body with initial stresses have the form

$$\partial\sigma_{ij} / \partial x_j = \varepsilon^{-1} f_i \text{ in } Q_\varepsilon \tag{1.1}$$

$$\sigma_{ij} n_j = g_i^\pm \text{ on } \Gamma_\varepsilon^\pm; \ \mathbf{u}^\varepsilon(\mathbf{x}) = 0 \text{ on } \Gamma_0$$

where Q_ε is the domain occupied by the membrane and $\Gamma_\varepsilon^\pm$, Γ_0 is its surface (Fig. 1).

The relation between the current stresses σ_{ij}, the strains $\mathbf{u}^\varepsilon$ and the initial stresses can be written in the form [10, 11, 18]

$$\sigma_{ij} = \varepsilon^{-1} \mathcal{A}_{ijkl} \partial u_k / \partial x_l \tag{1.2}$$

$$\mathcal{A}_{ijkl} = a_{ijkl}(\mathbf{x}/\varepsilon) + \sigma_{jl}^*(\tilde{\mathbf{x}}, \mathbf{x}/\varepsilon)\delta_{ik} \tag{1.3}$$

where a_{ijkl} are the components of the elasticity constants tensor, $\delta_{ik} = 1$, if $i = k$, and $\delta_{ik} = 0$ if $i \neq k$ and $\mathbf{x} = (x_1, x_2)$ are the slow variables in the plane of the membrane. The functions $a_{ijkl}(\mathbf{y})$, $\sigma_{jl}^*(\tilde{\mathbf{x}}, \mathbf{y})$ are periodic with respect to y_1 and y_2 with a PC S_1 (S_1 is the projection of the PC P_1 on the y_1y_2 plane (Fig. 1) in accordance with the periodic structure of the membrane.

We introduce the notation

$$\langle\cdot\rangle = \frac{1}{\operatorname{mes} S_1} \int_{P_1} (\cdot) dy, \ \langle\cdot\rangle_\gamma = \frac{1}{\operatorname{mes} S_1} \int_\gamma (\cdot) dy$$

(the first expression is an average over the PC $P_1 = \varepsilon^{-1}P_\varepsilon = \{\mathbf{y} = \mathbf{x}/\varepsilon\colon \mathbf{x} \in P_\varepsilon\}$ in the dimensionless (fast) variables $\mathbf{y} = \mathbf{x}/\varepsilon$ (Fig. 1) while the second is an average over the lateral surface γ of the PC P_1.

The derivative of a function of the form $f(\tilde{\mathbf{x}}, \mathbf{y})$ is calculated by replacing the differential operator according to the rule

$$\frac{\partial f}{\partial x_\alpha} \to f,_{\alpha x} + \varepsilon^{-1} f,_{\alpha y}, \frac{\partial f}{\partial x_3} \to \varepsilon^{-1} f_{,3y} \ (\alpha = 2,3) \tag{1.4}$$

$$(f_{,\alpha x} = \partial f / \partial x_\alpha, \ f_{,iy} = \partial f / \partial y_i)$$

Henceforth, Latin lower-case subscripts take the values 1, 2 and 3 while Greek subscripts and superscripts take the values 1 and 2, and $m = -1, 0, \ldots; n = 0, 1, \ldots$.

We shall seek a solution of problem (1.1)–(1.3) in a form which is analogous to that which has been used previously [2] but, in accordance with (1.2), we commence the expansion for the stresses σ_{ij} with a term of the order of ε^{-1}

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^{(0)}(\tilde{\mathbf{x}}) + \varepsilon\mathbf{u}^{(1)}(\tilde{\mathbf{x}}, \mathbf{y}) + \ldots = \sum_{n=0}^{\infty} \varepsilon^{n}\mathbf{u}^{(n)}(\tilde{\mathbf{x}}, \mathbf{y}) \tag{1.5}$$

$$\sigma_{ij} = \varepsilon^{-1}\sigma_{ij}^{(-1)}(\tilde{\mathbf{x}}, \mathbf{y}) + \ldots = \sum_{m=-1}^{\infty} \varepsilon^{m}\sigma_{ij}^{(m)}(\tilde{\mathbf{x}}, \mathbf{y}) \tag{1.6}$$

Substituting (1.5) into (1.2) and taking account of (1.4), we obtain a relation and, on equating the expressions accompanying ε in this relation, we find that

$$\sigma_{ij}^{(-1)} = \mathcal{A}_{ijkl}u_{k,ly}^{(1)} + \mathcal{A}_{ijk\alpha}u_{k,\alpha x}^{(0)} \tag{1.7}$$

The terms of expansion (1.6) satisfy the equations [2]

$$\sigma_{ij,jy}^{(m)} + \sigma_{i\alpha,\alpha x}^{(m-1)} = 0 \quad (\sigma_{ij}^{(-2)} \equiv 0) \tag{1.8}$$

which are obtained by substituting (1.6) into (1.1) and taking account of (1.4) (for more detail, see [2]). Here, it is only the case when $m = -1$ which is of interest. For this value, equality (1.8) gives

$$\sigma_{ij,jy}^{(-1)} = 0$$

On substituting (1.8) here, we arrive in the usual way [2] at the cellular problem

$$(\mathcal{A}_{ijkl}(\tilde{\mathbf{x}}, \mathbf{y})\mathcal{N}_{k,ly}^{p\alpha} + \mathcal{A}_{ijp\alpha}(\tilde{\mathbf{x}}, \mathbf{y}))_{,iy} = 0 \quad \text{in} \quad P_1 \tag{1.9}$$

$$(\mathcal{A}_{ijkl}(\tilde{\mathbf{x}}, \mathbf{y})\mathcal{N}_{k,ly}^{p\alpha} + \mathcal{A}_{ijp\alpha}(\tilde{\mathbf{x}}, \mathbf{y}))n_j = 0 \quad \text{on} \quad \gamma$$

The function $\mathcal{N}^{p\alpha}(\mathbf{y})$ is periodic in $y_1 y_2$ with a PC S_1.

As was stipulated above, suppose that σ_{ij}^{*} are determined from the solution of a problem in the theory of elasticity. Then, using (1.3), we have

$$(\mathcal{A}_{ijkl} - a_{ijkl})_{,jy} = \sigma_{jl,jy}^{*}\delta_{ik} = 0 \quad \text{in} \quad P_1 \tag{1.10}$$

$$(\mathcal{A}_{ijkl} - a_{ijkl})n_j = \sigma_{jl,}^{*}\, n_j\delta_{ik} = 0 \quad \text{on} \quad \gamma$$

Proposition 1. The equality

$$\mathcal{N}^{3\alpha}(\mathbf{y}) = -y_3\mathbf{e}_{\alpha} \tag{1.11}$$

holds.

In the case of (1.11), we have $(-\mathcal{A}_{ij\alpha 3} + \mathcal{A}_{ij3\alpha})_{,jy} = 0$ by virtue of the symmetry of a_{ijkl} and (1.10). Furthermore, $(-\mathcal{A}_{ij\alpha 3} + \mathcal{A}_{ij3\alpha})n_j = 0$ on γ by virtue of the symmetry of a_{ijkl} and (1.10).

When account is taken of Assumption 1, we obtain a representation of $\mathbf{u}^{(1)}$ in terms of the solution of the cellular problem (1.9)

$$\mathbf{u}^{(1)} = -y_3\mathbf{e}_{\alpha}u_{3,\alpha x}^{(0)}(\tilde{\mathbf{x}}) + \mathcal{N}^{\beta\alpha}(\mathbf{y})u_{\beta,\alpha x}^{(0)}(\tilde{\mathbf{x}}) + \mathbf{v}(\tilde{\mathbf{x}}) \tag{1.12}$$

This solution is identical in its form to the solution from [2] but the coefficients of the cellular problem (1.9) differ from the elasticity constants a_{ijkl} and depend on the initial stresses.

Substitution of (1.12) into (1.8) gives

$$\sigma_{ij}^{(-1)} = (-\mathcal{A}_{ij\alpha 3} + \mathcal{A}_{ij3\alpha})u_{3,\alpha x}^{(0)}(\tilde{\mathbf{x}}) + (\mathcal{A}_{ij\beta\alpha} + \mathcal{A}_{ijkl}\mathcal{N}_{k,ly}^{\beta\alpha})u_{\beta,\alpha x}^{(0)}(\tilde{\mathbf{x}}) \tag{1.13}$$

By making use of the definition of $\mathcal{A}_{ijkl}$ (1.3), we obtain from (1.13) that

$$\sigma_{ij}^{(-1)} = (-\sigma_{j3}^{*}\delta_{i\alpha} + \sigma_{j\alpha}^{*}\delta_{i3})u_{3,\alpha x}^{(0)}(\tilde{\mathbf{x}}) + (\mathcal{A}_{ij\beta\alpha} + \mathcal{A}_{ijkl}\mathcal{N}_{k,ly}^{\beta\alpha})u_{\beta,\alpha x}^{(0)}(\tilde{\mathbf{x}}) \tag{1.14}$$

In the case in question, the equilibrium stresses $N_{ij} = \langle\sigma_{ij}^{(-1)}\rangle$ are identical to those obtained in [2] and have the form (for a detailed derivation of the equilibrium equation for a membrane, see [1, 2, 6])

$$N_{\alpha\gamma,\gamma x} = \langle f_\alpha \rangle, \quad N_{3\gamma,3x} = \langle f_3 \rangle + \langle g_3^+ - g_3^- \rangle_\gamma \tag{1.15}$$

We now consider (1.14) for different values of the indices. In the case when $i = 3, j = \gamma$, Eq. (1.14), taking (1.3) into account, gives

$$N_{3\gamma} = \langle \sigma_{3\gamma}^{(-1)} \rangle = \langle \sigma_{\gamma\alpha}^{*} \rangle u_{3\alpha x}^{(0)}(\tilde{\mathbf{x}}) + \langle \mathcal{A}_{3\gamma\beta\alpha} + \mathcal{A}_{3\gamma kl} \mathcal{N}_{k,ly}^{\beta\alpha} \rangle u_{\beta,\alpha x}^{(0)}(\tilde{\mathbf{x}}) \tag{1.16}$$

For $i = \kappa, j = \gamma$, taking (1.3) into account. Eq. (1.14) gives

$$N_{\kappa\gamma} = \langle \sigma_{\kappa\gamma}^{(-1)} \rangle = -\langle \sigma_{\gamma 3}^{*} \rangle u_{3,\alpha x}^{(0)}(\tilde{x}) + \langle \mathcal{A}_{\kappa\gamma\beta\alpha} + \mathcal{A}_{\kappa\gamma kl} \mathcal{N}_{k,ly}^{\beta\alpha} \rangle u_{\beta,\alpha x}^{(0)}(\tilde{x}) \tag{1.17}$$

The cellular problem (1.9) is now considered. We multiply the equation from (1.9) by y_3 and integrate the result by parts over the PC P_1. When account is taken of the boundary condition, we obtain from (1.9) and the periodicity of $\mathcal{N}^{\beta\alpha}$ and y_3 with respect to y_1 and y_2, that

$$\langle \mathcal{A}_{i3kl} \mathcal{N}_{k,ly}^{\beta\alpha} + \mathcal{A}_{i3\beta\alpha} \rangle = 0 \tag{1.18}$$

Using (1.18), the symmetry of a_{ijkl} and the definition of $\mathcal{A}_{ijkl}$, the last term on the right-hand side of (1.16) can be rewritten in the form

$$\langle (\sigma_{3l}^{*} \delta_{\gamma k} - \sigma_{\gamma l}^{*} \delta_{3k}) \mathcal{N}_{k,ly}^{\alpha\beta} \rangle + \langle \sigma_{3\alpha}^{*} \rangle \delta_{\gamma\beta}$$

Proposition 2. If σ_{ij}^{*} are determined from the solution of a problem in the theory of elasticity then

$$\langle \sigma_{i3}^{*} \rangle = 0.$$

The proof of this is analogous to the proof of relation (1.18). As a result, (1.16) and (1.17) take the form

$$N_{3\gamma} = N_{\gamma\alpha}^{*} u_{3,\alpha x}^{(0)} + R_{\gamma\beta\alpha} u_{\beta,\alpha x}^{(0)} \tag{1.19}$$

$$N_{\kappa\gamma} = \hat{A}_{\kappa\gamma\beta\alpha} u_{\beta,\alpha x}^{(0)} \tag{1.20}$$

where

$$N_{\gamma\alpha}^{*} = \langle \sigma_{\gamma\alpha}^{*} \rangle, \quad R_{\gamma\beta\alpha} = \langle \sigma_{3l}^{*} \mathcal{N}_{\gamma,ly}^{\alpha\beta} \rangle - \langle \sigma_{\gamma l}^{*} \mathcal{N}_{3,ly}^{\alpha\beta} \rangle$$
$$\hat{A}_{\kappa\gamma\beta\alpha} = \langle \mathcal{A}_{\kappa\gamma\beta\alpha} + \mathcal{A}_{\kappa\gamma kl} \mathcal{N}_{k,ly}^{\beta\alpha} \rangle$$

Here, $N_{\gamma\alpha}^{*}$ are the initial stresses in the plane of the membrane and $\hat{A}_{\kappa\gamma\beta\alpha}$ are the averaged elasticity constants of the stressed body (which is two-dimensional in the case under consideration). The quantities $\hat{A}_{\kappa\gamma\beta\alpha}$, generally speaking, depend on the prior stresses in the plane of the membrane. This dependence is analogous to that found in [10–13].

The boundary conditions for the strains have the form

$$u_\alpha^{(0)}(\tilde{\mathbf{x}}) = 0 \quad (\alpha = 1,2) \text{ on } \partial S \tag{1.21}$$

$$u_3^{(0)}(\tilde{\mathbf{x}}) = 0 \quad \text{on } \partial S \tag{1.22}$$

The equilibrium equations (1.15) when $\langle f_\alpha \rangle = 0$ with constitutive equation (1.20) and boundary condition (1.21) have the solution $\mathbf{u}_\alpha(\tilde{\mathbf{x}}) = 0$ (subject to the condition that the initial stresses do not cause stability loss of the membrane as a plane body). The latter condition is nearly always satisfied in practice since the initial stresses are small compared with the elasticity constants [10, 11]. Then, (1.19) takes the form

$$N_{3\gamma} = N_{\gamma\alpha}^{*} u_{3,\alpha x}^{(0)} \tag{1.23}$$

with the equilibrium equation

$$N_{3\gamma,\gamma x} = \langle f_3 \rangle + \langle g_3^+ - g_3^- \rangle_\gamma \tag{1.24}$$

and boundary condition (1.22).

Equations (1.23) and (1.24) can be transformed into the classical equation for the flexure of a membrane

$$(N^*_{\gamma\alpha} u^{(0)}_{3,\alpha x})_{,\gamma x} = \langle f_3 \rangle + \langle g_3^+ - g_3^- \rangle_\gamma$$

The situation is different from the classical one when there are non-zero forces of the order of ε^{-1} in the plane of the membrane.

A homogeneous membrane. In the case of a homogeneous membrane $\mathcal{N}^{\alpha\beta}$ is solely a function of y_3 and $\sigma^*_{i3} = 0$, by virtue of which $R_{\gamma\beta\alpha} = 0, \hat{A}_{\kappa\gamma\beta\alpha} = \langle \mathcal{A}_{\kappa\gamma\beta\alpha} + \mathcal{A}_{\kappa\gamma k3} \mathcal{N}^{\beta\alpha}_{k,3y} \rangle$. Hence, Eqs (1.19) and (1.20) are not connected in this case.

Layered membranes. Suppose that a membrane is made up of layers of isotropic materials which are parallel to the x_1x_2 plane. Then, as in the preceding case, $\mathcal{N}^{\alpha\beta}$ is solely a function of y_3 and $\sigma^*_{i3} = 0$ and, by virtue of this, $R_{\gamma\beta\alpha} = 0$.

The general case. It is well known [2, 6] that the local stresses σ^*_{ij} are associated with the global strains **U** in the plane of an inhomogeneous body with a periodic structure by a formula of the type

$$\sigma^*_{ij} = (a_{ijAB} + a_{ijkl} N^{AB}_{k,ly}) U_{A,Bx} (A, B - 1, 2) \tag{1.25}$$

Substituting (1.25) into the formula for $R_{\gamma\beta\alpha}$, we obtain

$$R_{\gamma\beta\alpha} = [\langle a_{3lAB} \mathcal{N}^{\alpha\beta}_{\gamma,ly} \rangle + \langle a_{3lmn} N^{AB}_{m,ny} \mathcal{N}^{\alpha\beta}_{\gamma,ly} \rangle - \langle a_{\gamma lAB} \mathcal{N}^{\alpha\beta}_{3,ly} \rangle - \langle a_{\gamma lmn} N^{\alpha\beta}_{m,ny} \mathcal{N}^{\alpha\beta}_{3,ly} \rangle] U_{A,Bx}$$

In membranes made of isotropic materials, of the a_{3lAB} ($l = 1, 2, 3; A, B = 1, 2$) only a_{33AB} is non-zero and, of $a_{\gamma l AB}$ ($\gamma = 1, 2$) only $a_{\gamma\delta AB}$ is non-zero. Then, in the case in question

$$R_{\gamma\beta\alpha} = [\langle a_{33AA} \mathcal{N}^{\alpha\beta}_{\gamma,3y} \rangle + \langle a_{3lmn} N^{AB}_{m,ny} \mathcal{N}^{\alpha\beta}_{\gamma,ly} \rangle - \langle a_{\gamma\delta AB} \mathcal{N}^{\alpha\beta}_{3,\delta y} \rangle - \langle a_{\gamma lmn} N^{\alpha\beta}_{m,ny} \mathcal{N}^{\alpha\beta}_{3,ly} \rangle] U_{A,Bx}$$

2. A STRING

We now consider a domain of small diameter Q_ε which is obtained by periodic repetition of a certain periodic cell (PC) along the x_1 axis (Fig. 2). The characteristic size of this PC (which is identical to the characteristic diameter of the string) $\varepsilon \ll 1$. When $\varepsilon \to 0$, the domain Q_ε contracts to an interval [–1, 1] on the x_1 axis. The stresses σ^*_{ij} in the material, which occupies the domain Q_ε, are determined from the solution of a problem in the theory of elasticity.

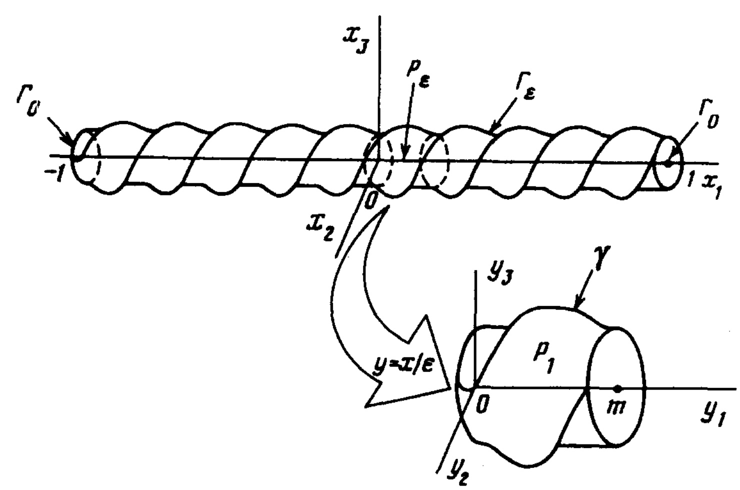

Fig. 2.

We shall assume that the elasticity constants of the string material a_{ijkl} and the initial stresses σ^*_{ij} are of the same order of magnitude ε^{-2}.

Again, we take the orders of magnitude to be identical although, in classical theory, the stiffnesses are neglected compared with the stresses. The equilibrium equations for the body then have the form of (1.1)–(1.3) [7], where Q_ε is the domain occupied by the string, and Γ_ε and Γ_0 is its surface (Fig. 2).

In this case, the relation between the actual stresses σ_{ij}, the strains $\mathbf{u}^\varepsilon$ and the initial stresses can be written in the form [10, 11, 18]

$$\sigma_{ij} = \varepsilon^{-2}\mathcal{A}_{ijkl}(x_1, \mathbf{y})\partial u_k / \partial x_l \tag{2.1}$$

$$\mathcal{A}_{ijkl} = a_{ijkl}(\mathbf{y}) + \sigma^*_{jl}(x_1, \mathbf{y})\delta_{ik} \tag{2.2}$$

where x_1 is a slow variable along the string axis. The functions $a_{ijkl}(\mathbf{y})$, $\sigma^*_{jl}(x_1, \mathbf{y})$ are periodic with respect to y_1 with period m (Fig. 2) in accordance with the periodic structure of the string.

We use the notation

$$\langle\cdot\rangle = \frac{1}{m}\int_{P_1} (\cdot)d\mathbf{y}, \quad \langle\cdot\rangle_\gamma = \frac{1}{m}\int_{P_\gamma} (\cdot)d\mathbf{y}$$

to denote the average over the PC $P_1 = \varepsilon^{-1}P_\varepsilon = \{\mathbf{y} = \mathbf{x}/\varepsilon : \mathbf{x} \in P_\varepsilon\}$ in the dimensionless (fast) variables $\mathbf{y} = \mathbf{x}/\varepsilon$ (Fig. 2) and the average over the lateral surface γ of the PC P_1.

The derivative of a function of the form $f(x_1, \mathbf{y})$ is calculated by replacing the differential operator according to the rule [19]

$$\frac{\partial f}{\partial x_1} \to f_{,1x} + \varepsilon^{-1} f_{,1y}, \quad \frac{\partial f}{\partial x_\alpha} \to \varepsilon^{-1} f_{,\alpha y} (\alpha = 2, 3) \tag{2.3}$$

$$(f_{,iy} = \partial f / \partial y_i, \ f_{,1x} = \partial f / \partial x_1)$$

Henceforth, Latin subscripts take the value 1, 2 and 3 and Greek subscripts take the values of 2 and 3, and $m = -2, -1, \ldots; n = 0, 1, \ldots$.

We shall seek a solution of problem (1.1)–(1.3) in a form which is analogous to that used previously [4] but, in accordance with (2.2), we shall commence the expansion for the stresses σ_{ij} with a term of the order of ε^{-2}

$$\mathbf{u}^\varepsilon = \mathbf{u}^{(0)}(x_1) + \varepsilon\mathbf{u}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{u}^{(n)}(x_1, \mathbf{y}) \tag{2.4}$$

$$\sigma_{ij} = \varepsilon^{-2}\sigma_{ij}^{(-2)}(x_1, \mathbf{y}) + \ldots = \sum_{m=-2}^{\infty} \varepsilon^m \sigma_{ij}^{(m)}(x_1, \mathbf{y}) \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.1) and taking account of (2.3), we obtain

$$\sum_{m=-2}^{\infty} \varepsilon^m \sigma_{ij}^{(m)} = \sum_{n=0}^{\infty} \varepsilon^{n-3}\mathcal{A}_{ijkl}u_{k,ly}^{(n)} + \varepsilon^{n-2}\mathcal{A}_{ijk1}u_{k,1x}^{(n)} \tag{2.6}$$

On equating the expressions accompanying ε^{-2} in (2.6), we obtain

$$\sigma_{ij}^{(-2)} = \mathcal{A}_{ijkl}(x_1, \mathbf{y})u_{k,ly}^{(1)} + \mathcal{A}_{ijk1}(x_1, \mathbf{y})u_{k,1x}^{(0)} \tag{2.7}$$

The terms of expansion (2.5) satisfy the equations [4]

$$\sigma_{ij,iy}^{(m)} + \sigma_{i1,1x}^{(m-1)} = 0 \text{ in } P_1, \quad \sigma_{ij}^{(m)} n_j = 0 \text{ on } \gamma \tag{2.8}$$

which are obtained by substituting (2.5) into (1.1) and taking account of (2.3) (for more detail, see [4, 6]). Here γ is the lateral surface of the PC P_1, and we are only interested in the case when $m = -2$. In this case (2.8) gives

$$\sigma_{ij,jy}^{(-2)} = 0 \text{ в } P_1, \quad \sigma_{ij}^{(-2)} n_j = 0 \text{ on } \gamma \tag{2.9}$$

Substituting (2.7) here, we arrive at the cellular problem

$$\begin{aligned} &(\mathcal{A}_{ijkl}(x_1, y)\mathcal{N}_{k,ly}^p + \mathcal{A}_{ijp1}(x_1, \mathbf{y}))_{,jy} = 0 \text{ in } P_1 \\ &(\mathcal{A}_{ijkl}(x_1, \mathbf{y})\mathcal{N}_{k,ly}^p + \mathcal{A}_{ijp1}(x_1, \mathbf{y}))n_j = 0 \text{ on } \gamma \, \mathbf{N}^p(\mathbf{y}) \end{aligned} \tag{2.10}$$

which is periodic in y_1 with period m.

Suppose that the stresses σ_{ij}^*, as was stipulated above, are determined from the solution of a problem in the theory of elasticity. Then, (1.10) is satisfied in the case of these stresses.

Proposition 3.

$$\mathbf{N}^\alpha(\mathbf{y}) = -y_\alpha \mathbf{e}_1 \tag{2.11}$$

For (2.11), we have $(-\mathcal{A}_{ij1\alpha} + \mathcal{A}_{ij\alpha1})_{,jy} = 0$ by virtue of the symmetry of a_{ijkl} and (1.10). Furthermore, $(-\mathcal{A}_{ij1\alpha} + \mathcal{A}_{ij\alpha1})n_j = 0$ in γ by virtue of the symmetry of a_{ijkl} and (2.10).

When account is taken of Proposition 3, we obtain a representation of $\mathbf{u}^{(1)}$ in terms of the solution of the cellular problem (2.10)

$$\mathbf{u}^{(1)} = -y_\alpha \mathbf{e}_1 u_{\alpha,1x}^{(0)}(x_1) + \mathbf{N}^1(\mathbf{y}) u_{1,1x}^{(0)}(x_1) \tag{2.12}$$

This solution is identical in for to the solution from [4], but the coefficients of the cellular problem (2.10) differ from the elasticity constants a_{ijkl} and depend on the initial stresses.

Substituting expressions (2.12) into (2.7) and using the definition of $\mathcal{A}_{ijkl}$ (2.2), we obtain

$$\sigma_{ij}^{(-2)} = (-\sigma_{j\alpha}^*\delta_{i1} + \sigma_{j1}^*\delta_{i\alpha}) u_{\alpha,1x}^{(0)}(x_1) + (\mathcal{A}_{ij11} + \mathcal{A}_{ijkl}\mathcal{N}_{k,ly}^1) u_{1,1x}^{(0)}(x_1) \tag{2.13}$$

In the case under consideration, the equilibrium equations for the stresses $N_i = \langle \sigma_{i1}^{(-2)} \rangle$ are identical to those obtained previously [4]

$$N_{1,1x} = \langle f_1 \rangle, \quad N_{\alpha,1x} = \langle f_\alpha \rangle + \langle g_\alpha \rangle_\gamma \tag{2.14}$$

Equalities (2.13) enable us to obtain expressions for the stresses in terms of the deformation characteristics.

We will now consider equalities (2.13) for different values of the subscripts while taking equality (2.2) into consideration. We have

$$N_1 = \langle \sigma_{11}^{(-2)} \rangle = \langle \mathcal{A}_{1111} + \mathcal{A}_{11kl}\mathcal{N}_{k,ly}^1 \rangle u_{1,1x}^{(0)} - \langle \sigma_{1\alpha}^* \rangle u_{\alpha,1x}^{(0)} \quad (i=1) \tag{2.15}$$

$$N_\beta = \langle \sigma_{\beta1}^{(-2)} \rangle = \langle \sigma_{11}^* \rangle u_{\beta,1x}^{(0)} + \langle \mathcal{A}_{\beta111} + \mathcal{A}_{\beta1kl}\mathcal{N}_{k,ly}^1 \rangle u_{1,1x}^{(0)} \quad (i=\beta) \tag{2.16}$$

Proposition 4. If σ_{ij}^* are determined from the solution of a problem in the theory of elasticity, then $\langle \sigma_{1\alpha}^* \rangle = 0$.

In order to verify this, we multiply the equilibrium equation $\sigma_{ij,jy}^* = 0$ by y_α and integrate the result over the PC P_1, taking account of the boundary condition $\sigma_{ij}^* n_j = 0$ on γ and the periodicity of all the functions (including y_α) with respect to y_1.

As a result, (2.15) and (2.16) take the form

$$\begin{aligned} &N_1 = \hat{A} u_{1,1x}^{(0)}, \quad N_\beta = N_1^* u_{\beta,1x} + R_\beta u_{1,1x}^{(0)} \\ &\hat{A} = \langle \mathcal{A}_{1111} + \mathcal{A}_{11kl}\mathcal{N}_{k,ly}^1 \rangle, \quad N_1^* = \langle \sigma_{11}^* \rangle \\ &R_\beta = \langle \mathcal{A}_{\beta111} + \mathcal{A}_{\beta1kl}\mathcal{N}_{k,ly}^1 \rangle \end{aligned} \tag{2.17}$$

Here N_1^* is the initial axial stress of the string, $\hat{A}$ are the averaged elasticity constants of the stressed body (which is one-dimensional in the case in question). The quantity $\hat{A}$, generally speaking, depends of the prior stress in the string. This dependence is analogous to that found previously in [10–13].

The boundary conditions for the strains have the form

$$u_\alpha^{(0)}(\pm 1) = 0, \quad u_1^{(0)}(\pm 1) = 0 \tag{2.18}$$

The equilibrium equations when $\langle f_1 \rangle = 0$ with the constitutive equations (2.15) and boundary condition (2.18) have the solution $u_1^{(0)}(x_1) \equiv 0$ (subject to the condition that the initial stresses σ_{ij}^* do not cause any stability loss of the string as a one-dimensional body, that is, as a rod). The last condition is always satisfied in practice since the initial stresses are small compared with the elasticity constants [10, 11]. Then, (2.17) takes the form

$$N_\beta = N_1^* u_{\beta,1x}^{(0)} \tag{2.19}$$

with the equilibrium equation

$$N_{\beta,1x} = \langle f_\beta \rangle + \langle g_\beta \rangle_\gamma \tag{2.20}$$

and boundary condition (2.18).

Equations (2.19) and (2.20) can be transformed into the classical equation of a string

$$(N_1^* u_{\beta,1x}^{(0)})_{,1x} = \langle f_\beta \rangle + \langle g_\beta \rangle_\gamma$$

The situation is different from the classical one when there are non-zero forces of the order of ε^{-2} acting along the x_1 axis.

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